BAULKHAM HILLS

2017 HSC
Trial Examination

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A reference sheet is provided at the back of this paper
- In Questions 11 – 14, show relevant mathematical reasoning and/or calculations

Total marks – 70

Section I Pages 2 – 4

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II Pages 5 – 10

60 marks

- Attempt Questions 11 – 14
- Allow about 1 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer page in the writing booklet for Questions 1 – 10.

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1 How many arrangements of the word **GEOMETRY** are possible if the letters **T** and **R** are to be together?

(A) $2 \times 6!$ (B) $2 \times 7!$ (C) $7!$ (D) $\frac{7!}{2!}$

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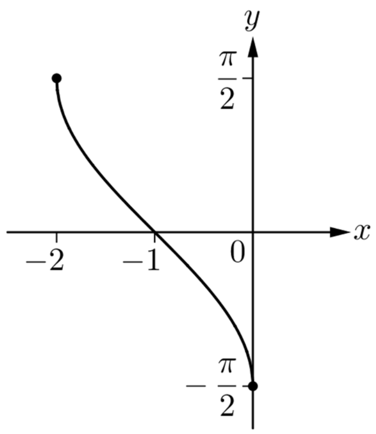
2 $\frac{\sin 2\phi + \sin \phi}{\cos 2\phi + \cos \phi + 1}$ is equivalent to:

(A) $\cot \phi$ (B) $\sec \phi$ (C) $\sin \phi$ (D) $\tan \phi$

3



The diagram above shows the graph of:

(A) $y = \sin^{-1}(x+1)$ (B) $y = \sin^{-1}(x-1)$

(C) $y = \cos^{-1}(x+1) - \frac{\pi}{2}$ (D) $y = \cos^{-1}(x-1) - \frac{\pi}{2}$

4 Given that α and β are both acute angles, evaluate $\sin(\alpha+\beta)$ if $\sin\alpha = \dfrac{8}{17}$ and $\sin\beta = \dfrac{4}{5}$.

(A) $\dfrac{108}{85}$ (B) $\dfrac{84}{85}$ (C) $\dfrac{36}{85}$ (D) $\dfrac{28}{85}$

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5 The exact value of $\sin^{-1}\left(\sin\dfrac{4\pi}{3}\right)$ is:

(A) $\dfrac{4\pi}{3}$ (B) $\dfrac{\pi}{3}$ (C) $-\dfrac{\pi}{3}$ (D) $-\dfrac{4\pi}{3}$

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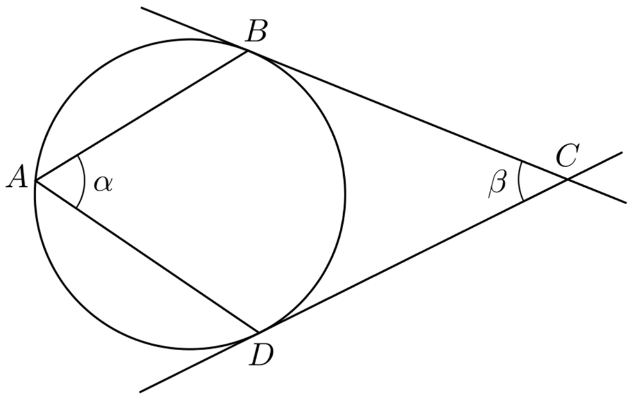
6 In the diagram below, *BC* and *DC* are tangents to the circle at *B* and *D* respectively.



Which of the following statements is correct?

(A) $2\alpha + \beta = 180°$ (B) $\alpha + \beta = 180°$

(C) $2\alpha - \beta = 180°$ (D) $\alpha + 2\beta = 180°$

7 The roots of the equation $x^3 - 5x + 6 = 0$ are α , β and γ .

The value of $\alpha + \beta + \gamma$ and the value of $\alpha\beta\gamma$ are respectively:

(A) -5 and -6 (B) 5 and -6 (C) 0 and 6 (D) 0 and -6

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8 A particle moves under simple harmonic motion such that its position *x* metres after *t* seconds is given by $x = 8\sin\left(\frac{t}{4} - \frac{\pi}{2}\right)$.

Which of the following statements is **<u>FALSE</u>**?

(A) The maximum speed of the particle is 2 m/s.

(B) The maximum acceleration of the particle is 0.5 m/s².

(C) The particle takes 8π seconds to travel between the extremities of its motion.

(D) The particle is initially left of the origin.

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9 Given $y = \cos^{-1}\left(\frac{1}{x}\right)$, the correct expression for $\frac{dy}{dx}$ is:

(A) $\frac{1}{\sqrt{x^2 - 1}}$ (B) $\frac{1}{x\sqrt{x^2 - 1}}$ (C) $-\frac{1}{\sqrt{x^2 - 1}}$ (D) $-\frac{1}{x\sqrt{x^2 - 1}}$

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10 The solution to $\ln\left(x^3 + 19\right) = 3\ln\left(x + 1\right)$ is:

(A) $x = 2$ (B) $x = -3$

(C) $x = -3$ or $x = 2$ (D) $x = -2$ or $x = 3$

End of Section I

Section II

60 marks
Attempt Questions 11 – 14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate section of the writing booklet. Extra writing paper is available.

In Questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 section of the writing booklet.

(a) Find $\int \sin^2\left(\frac{x}{2}\right)dx$. **2**

(b) $T\left(2t, t^2\right)$ is a point on the parabola $x^2 = 4y$ with focus S. The point P divides ST internally in a ratio of $1:2$.

(i) Find the coordinates of P in terms of t. **2**

(ii) Hence show that as T moves on the parabola $x^2 = 4y$, the locus of P is the parabola $9x^2 = 12y - 8$. **2**

(c) Using the substitution $u^2 = x + 1$, where $u > 0$, to find $\int_0^3 \frac{x+2}{\sqrt{x+1}}dx$. **3**

(d) Solve $x + 2 \le \frac{4}{x-1}$. **3**

Question 11 continues over the page

Question 11 (continued)

(e) Consider the curve with equation $y = e^{\sin x}$.

(i) Show that the tangent to the curve at the point where $x = \pi$ has equation $x + y - \pi - 1 = 0$. **2**

(ii) Find the acute angle between the tangent in part (i) and the line $y = -\frac{5}{2}x + 5$. **1**

Give your answer to the nearest minute.

End of Question 11

Question 12 (15 marks) Use the Question 12 section of the writing booklet.

(a)

(i) Show that the graph of $f(x) = x^5 + 2x - 20$ has only one x-intercept. **2**

(ii) Confirm that $f(x)$ has a real root between $x = 1$ and $x = 2$. **1**

(iii) Starting with $x = 1.5$, use one application of Newton's method to find a better approximation for this root. Write your answer correct to 2 decimal places. **1**

(b) A particle moves along the x-axis according to the equation $x = 6\sin 2t - 2\sqrt{3}\cos 2t$, where x is in metres and t is in seconds.

(i) Express x in the form $R\sin(2t - \alpha)$ where $R > 0$ and $0 \leq \alpha \leq \dfrac{\pi}{2}$. **2**

(ii) Show that the particle is moving in simple harmonic motion. **2**

(iii) State the period of this particle's motion. **1**

(iv) When is the first time that the particle is 6 metres right of the origin? **2**

(c) Bobby has a can of lemonade at a temperature of 23°C. He places this can in a fridge set at a constant temperature of 3°C.

After t minutes, the temperature, c (in °C), of the can of lemonade satisfies the equation

$$\frac{dc}{dt} = -\frac{1}{25}(c - 3).$$

(i) Show that $c = 3 + ae^{-\frac{t}{25}}$ satisfies the above differential equation, where a is a constant. **1**

(ii) Bobby would like to drink the can of lemonade when its temperature is 5°C. If he puts the can in the fridge at 8:50 a.m., what is the earliest time that he should drink the can of lemonade? Give your answer to the nearest minute. **3**

End of Question 12

Question 13 (15 marks) Use the Question 13 section of the writing booklet.

(a) A group of 10 people, consisting of 6 girls and 4 boys, decided to go to the movies where they sit together in the same row. How many ways can the 10 people be seated in a row if:

(i) there are no restrictions? **1**

(ii) the 4 boys are all seated together? **2**

(iii) at least one of the boys is separated from the other boys? **1**

(b) Consider the equation $y = \sqrt{1-x^2} + x\sin^{-1} x$.

(i) Find the expression for $\dfrac{dy}{dx}$. **2**

(ii) Hence, or otherwise, evaluate $\int_0^1 \sin^{-1} x \, dx$. **1**

(c) A polynomial $f(x)$ is given by the equation $f(x) = x(x+1) - a(a+1)$ for some constant a.

(i) Use the remainder theorem to find one factor of $f(x)$. **1**

(ii) By division, or otherwise, express $f(x)$ as a product of linear factors. **1**

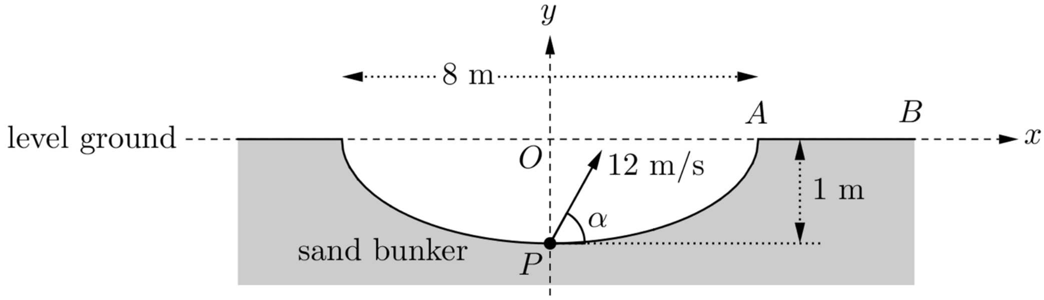
(d) Prove by mathematical induction that $4^n + 5^n + 6^n$ is divisible by 15 for all odd integers $n \geq 1$. **3**

(e) Find the term independent of x in the expansion of $\left(\dfrac{3x^2}{2} - \dfrac{1}{3x}\right)^9$. **3**

End of Question 13

Question 14 (15 marks) Use the Question 14 section of the writing booklet.

(a) The diagram below shows the cross-section of a sand bunker on a golf course. A golf ball is lying at point *P*, at the middle of the bottom of the sand bunker. The sand bunker is 8 metres wide and 1 metre deep at its deepest point, and is surrounded by level ground. The point *A* is at the edge of the bunker, and the line *AB* lies on level ground.



The golf ball is hit towards *A* with an initial speed of 12 metres per second, at an angle of elevation of α . The acceleration due to gravity is taken as 10 m/s².

It can be shown that the golf ball's trajectory at time *t* seconds after being hit is defined by the equations:

$$x = 12t\cos\alpha \text{ and } y = -5t^2 + 12t\sin\alpha - 1 \quad \text{(Do NOT prove these.)}$$

where *x* and *y* are the horizontal and vertical displacements (in metres) of the golf ball from the origin *O* shown in the diagram.

(i) Given $\alpha = 30°$, how far right of *A* will the golf ball land? **2**

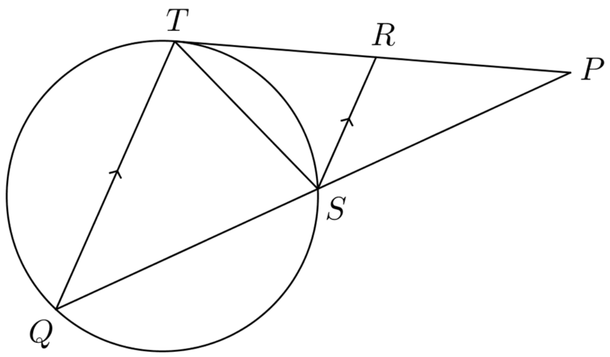
(ii) Find the maximum height above level ground reached by the ball if $\alpha = 30°$. **2**

(iii) Find the range of values of α , to the nearest minute, at which the golf ball must be hit so that it will land on level ground to the right of *A*. **3**

Question 14 continues over the page

Question 14 (continued)

(b) Triangle QST is inscribed in a circle. The tangent to the circle at T meets QS produced at P. The line through S parallel to QT meets PT at R.



(i) Show that $\Delta PST \parallel\!| \Delta PRS$. **2**

(ii) Hence show that $PT = \dfrac{ST \times PS}{RS}$. **1**

(c)

(i) Using the substitution $u = \cos x$, show that, for any constant k: **2**

$$\int_0^{\frac{\pi}{2}} (\cos x)^{2k} \sin x \, dx = \frac{1}{2k+1}.$$

(ii) By noting that $(\sin x)^{2n+1} = (\sin x)(1 - \cos^2 x)^n$, show using the binomial theorem that, for all positive integers n: **2**

$$\int_0^{\frac{\pi}{2}} (\sin x)^{2n+1} \, dx = \sum_{r=0}^{n} (-1)^r \, {}^nC_r \left(\frac{1}{2r+1} \right).$$

(iii) Use the result in part (ii) to evaluate $\int_0^{\frac{\pi}{2}} (\sin x)^5 \, dx$. **1**

End of Paper

YEAR 12 TRIAL EXAMINATION 2017
MATHEMATICS EXTENSION 1
MARKING GUIDELINES

Section I

Multiple-choice Answer Key

Question	Answer
1	C
2	D
3	C
4	B
5	C

Question	Answer
6	A
7	D
8	C
9	B
10	A

Questions 1 – 10

Sample solution

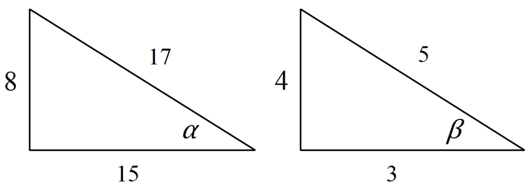
1. 2 ways of arranging **T** and **R**, $\frac{7!}{2!}$ ways of arranging the group of 2 letters and the remaining letters (including the repeated **E**'s).

$$\begin{aligned}\text{Number of arrangements} &= 2\times\frac{7!}{2!}\\ &= 7!\end{aligned}$$

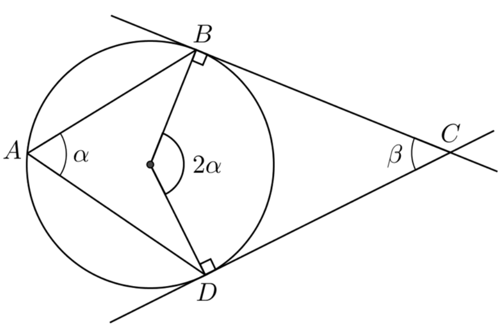
2.
$$\begin{aligned}\frac{\sin 2\phi+\sin\phi}{\cos 2\phi+\cos\phi+1} &= \frac{2\sin\phi\cos\phi+\sin\phi}{2\cos^2\phi-1+\cos\phi+1}\\ &= \frac{\sin\phi\left(2\cos\phi+1\right)}{\cos\phi\left(2\cos\phi+1\right)}\\ &= \tan\phi\end{aligned}$$

3. Simple translations of known graphs.

4.
$$\begin{aligned}\sin\left(\alpha+\beta\right) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta\\ &= \frac{8}{17}\times\frac{3}{5}+\frac{15}{17}\times\frac{4}{5}\\ &= \frac{24}{85}+\frac{60}{85}\\ &= \frac{84}{85}\end{aligned}$$



5.
$$\begin{aligned}\sin^{-1}\left(\sin\frac{4\pi}{3}\right) &= \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\\ &= -\frac{\pi}{3}\quad\left(\text{since } -\frac{\pi}{2}\le\sin^{-1}x\le\frac{\pi}{2}\right)\end{aligned}$$

6.


Construct the angle subtended at the centre to yield 2α as shown, then:

$$\begin{aligned}2\alpha+\beta+90°+90° &= 360°\quad\left(\text{angle sum of a quadrilateral} = 360°\right)\\ 2\alpha+\beta &= 180°\end{aligned}$$

7.	$\alpha+\beta+\gamma=-\frac{b}{a}$ $=\frac{0}{1}$ $=0$	$\alpha\beta\gamma=-\frac{d}{a}$ $=-\frac{6}{1}$ $=-6$
8.	$x=8\sin\left(\frac{t}{4}-\frac{\pi}{2}\right)$ $v=2\cos\left(\frac{t}{4}-\frac{\pi}{2}\right)$ $a=-\frac{1}{2}\sin\left(\frac{t}{4}-\frac{\pi}{2}\right)$ $=-\frac{1}{16}\times 8\sin\left(\frac{t}{4}-\frac{\pi}{2}\right)$ $=-\left(\frac{1}{4}\right)^2 x$	$T=\frac{2\pi}{n}$ $=\frac{2\pi}{\left(\frac{1}{4}\right)}$ $=8\pi$ Therefore, it only takes 4π seconds to travel from one extremity to the other.
9.	$u=\frac{1}{x}$ $\frac{du}{dx}=-\frac{1}{x^2}$ $y=\cos^{-1}(u)$ $\frac{dy}{du}=-\frac{1}{\sqrt{1-u^2}}$	$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$ $=-\frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}\times\frac{-1}{x^2}$ $=\frac{1}{\sqrt{x^4\left(1-\frac{1}{x^2}\right)}}$ $=\frac{1}{\sqrt{x^4-x^2}}$ $=\frac{1}{\sqrt{x^2\left(x^2-1\right)}}$ $=\frac{1}{x\sqrt{x^2-1}}$
10.	$\ln\left(x^3+19\right)=3\ln\left(x+1\right)$ $\ln\left(x^3+19\right)=\ln\left(x+1\right)^3$ $x^3+19=x^3+3x^2+3x+1$ $0=3x^2+3x-18$ $0=x^2+x-6$ $0=(x+3)(x-2)$ $\therefore x=2$ (as $x=-3$ lies outside of the natural domain of the logarithmic functions given in the question)	

Section II

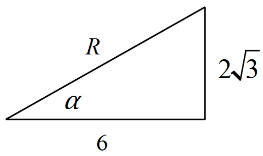
Question 11

	Sample solution	Suggested marking criteria
(a)	$\int \sin^2\left(\frac{x}{2}\right)dx = \int \frac{1}{2}\left(1-\cos x\right)dx$ $= \frac{1}{2}\left(x - \sin x\right) + c$	• 2 – correct solution • 1 – uses an appropriate trigonometric identity
(b)	(i) $S = (0,\ 1)$ $S(0,\ 1) \qquad T\left(2t, t^2\right)$ $1 : 2$ $P = \left(\frac{2\times 0 + 1\times 2t}{3}, \frac{2\times 1 + 1\times t^2}{3}\right)$ $= \left(\frac{2t}{3}, \frac{2+t^2}{3}\right)$	• 2 – correct solution • 1 – finds the coordinates of S
	(ii) $x = \frac{2t}{3} \quad \Rightarrow \quad t = \frac{3x}{2}$ $y = \frac{2+\left(\frac{3x}{2}\right)^2}{3}$ $= \frac{2+\frac{9x^2}{4}}{3}$ $= \frac{8+9x^2}{12}$ $12y = 8 + 9x^2$ $12y - 8 = 9x^2$	• 2 – correct solution • 1 – attempts to eliminate the parameter
(c)	$\int_0^3 \frac{x+2}{\sqrt{x+1}}dx = \int_1^2 \frac{u^2+1}{\not{u}} \times 2\not{u}\ du$ $= 2\left[\frac{u^3}{3} + u\right]_1^2$ $= 2\times\left[\left(\frac{2^3}{3}+2\right) - \left(\frac{1^3}{3}+1\right)\right]$ $= \frac{20}{3}$ $u^2 = x+1$ $2u = \frac{dx}{du}$ $2udu = dx$ When $x = 0, u = 1$ and when $x = 3, u = 2$.	• 3 – correct solution • 2 – correct integration • 1 – attempts to switch variables using the given substitution

Question 11 (continued)

Sample solution		Suggested marking criteria
(d)	$x+2 \le \dfrac{4}{x-1}$ $(x+2)(x-1)^2 \le 4(x-1)$ $(x-1)\left[(x+2)(x-1)-4\right] \le 0$ $(x-1)\left(x^2+x-6\right) \le 0$ $(x-1)(x+3)(x-2) \le 0$ $\therefore x \le -3$ or $1 < x \le 2 \quad (\text{as } x \ne 1)$	• 3 – correct solution • 2 – obtains the correct critical points • 1 – attempts to solve the inequation with an appropriate method
(e)	(i) $y = e^{\sin x}$ $\dfrac{dy}{dx} = \cos x e^{\sin x}$ When $x = \pi$: $y = e^{\sin \pi} = e^0 = 1$ $\dfrac{dy}{dx} = \cos \pi \, e^{\sin \pi} = -1e^0 = -1$ $y - y_1 = m(x - x_1)$ $y - 1 = -1(x - \pi)$ $y - 1 = -x + \pi$ $x + y - \pi - 1 = 0$	• 2 – correct solution • 1 – obtains the point or the gradient of the tangent at $x = \pi$
	(ii) $m_T = -1$ $\dfrac{x}{2} + \dfrac{y}{5} = 1$ $\dfrac{y}{5} = -\dfrac{x}{2} + 1$ $y = -\dfrac{5}{2}x + \dfrac{1}{5}$ $\therefore m_2 = -\dfrac{5}{2}$ $\tan\theta = \left\lvert \dfrac{-1-\left(-\frac{5}{2}\right)}{1+(-1)\times\left(-\frac{5}{2}\right)} \right\rvert = \dfrac{3}{7}$ $\theta = \tan^{-1}\left(\dfrac{3}{7}\right) = 23°12'$ (nearest minute)	• 1 – correct solution

Question 12

Sample solution			Suggested marking criteria
(a)	(i)	$f(x)=x^5+2x-20$ $f'(x)=5x^4+2$ >0 $f(0)=0^5+2\times0-20$ $=-20$ <0 $f(10)=10^5+2\times10-20$ $=10000$ >0 $f(x)$ is a continuous function, with $f(0)<0$ and $f(10)>0$; since $f'(x)>0$ over the entire domain of $f(x)$, $f(x)$ is a monotonically increasing function, and would therefore only cross the *x*-axis once.	• 2 – correct solution – uses a graphical method with correct explanation • 1 – correct expression for $f'(x)$ – attempts to use a graphical method
	(ii)	$f(1)=1^5+2\times1-20$ $=-17$ $f(2)=2^5+2\times2-20$ $=16$ Since $f(x)$ is a continuous function, with $f(1)<0$ and $f(2)>0$, therefore there exists a real root between $x=1$ and $x=2$.	• 1 – correct solution
	(iii)	$x_1=x_0-\dfrac{f(x_0)}{f'(x_0)}$ $=1.5-\dfrac{1.5^5+2\times1.5-20}{5\times1.5^4+2}$ $=1.84\ (2\text{ d.p.})$	• 1 – correct solution – correct substitution into Newton's approximation formula
(b)	(i)	$6\sin 2t-2\sqrt{3}\cos 2t\equiv R\sin(2t-\alpha)$ $=R\sin 2t\cos\alpha-R\cos 2t\sin\alpha$ $\therefore R\cos\alpha=6$ and $R\sin\alpha=2\sqrt{3}$  $R^2=6^2+\left(2\sqrt{3}\right)^2$ $=36+12$ $=48$ $R=4\sqrt{3}\quad(R>0)$ $\tan\alpha=\dfrac{2\sqrt{3}}{6}$ $=\dfrac{1}{\sqrt{3}}$ $\alpha=\dfrac{\pi}{6}$ $\therefore x=4\sqrt{3}\sin\left(2t-\dfrac{\pi}{6}\right)$	• 2 – correct solution • 1 – finds either *R* or α – expresses *x* in the correct form
	(ii)	$x=4\sqrt{3}\sin\left(2t-\dfrac{\pi}{6}\right)$ $v=8\sqrt{3}\cos\left(2t-\dfrac{\pi}{6}\right)$ $a=-16\sqrt{3}\sin\left(2t-\dfrac{\pi}{6}\right)$ $=-4x$ $=-2^2x$ Since acceleration is in the form $-n^2x$, therefore the particle is moving in simple harmonic motion.	• 2 – correct solution • 1 – correct expression for *a*

Question 12 (continued)

Sample solution			Suggested marking criteria
(b)	(iii)	$T=\frac{2\pi}{n}$ $=\frac{2\pi}{2}$ $=\pi$ seconds	• 1 – correct solution
	(iv)	$4\sqrt{3}\sin\left(2t-\frac{\pi}{6}\right)=6$ $\sin\left(2t-\frac{\pi}{6}\right)=\frac{6}{4\sqrt{3}}$ $=\frac{\sqrt{3}}{2}$ $2t-\frac{\pi}{6}=\frac{\pi}{3}$ $2t=\frac{\pi}{2}$ $t=\frac{\pi}{4}$ Therefore, the particle first reached 6 metres right of the origin at $\frac{\pi}{4}$ seconds.	• 2 – correct solution • 1 – attempts to solve $x=6$
(c)	(i)	$c=3+ae^{-\frac{t}{25}}$ $\frac{dc}{dt}=-\frac{1}{25}ae^{-\frac{t}{25}}$ $=-\frac{1}{25}\left(3+ae^{-\frac{t}{25}}-3\right)$ $=-\frac{1}{25}(c-3)$	• 1 – correct solution
	(ii)	When $t=0$, $c=23$: $c=3+ae^{-\frac{t}{25}}$ $23=3+ae^{0}$ $20=a$ $\therefore c=3+20e^{-\frac{t}{25}}$ $c=5$ when: $5=3+20e^{-\frac{t}{25}}$ $2=20e^{-\frac{t}{25}}$ $\frac{1}{10}=e^{-\frac{t}{25}}$ $10=e^{\frac{t}{25}}$ $\ln 10=\frac{t}{25}$ $t=25\ln 10$ $=58$ minutes (nearest minute) Therefore, Bobby should drink the can at 9:48am.	• 3 – correct solution • 2 – finds the time taken for the can to cool down to 5°C • 1 – finds the value of a

Question 13

Sample solution			Suggested marking criteria
(a)	(i)	$10! = 3\ 628\ 800$	• 1 – correct answer, or equivalent expression
	(ii)	$4! \times 7! = 120\ 960$	• 2 – correct answer, or equivalent expression • 1 – recognises the numerical implication of the given condition
	(iii)	At least one of the boys is separated implies the boys cannot be seated together, therefore, the number of ways the 10 people can sit is: $3\ 628\ 800 - 120\ 960 = 3\ 507\ 840$ *(leaving it as a subtraction is ok)*	• 1 – correct solution
(b)	(i)	$y = \sqrt{1-x^2} + x\sin^{-1}x$ $= \left(1-x^2\right)^{\frac{1}{2}} + x\sin^{-1}x$ $\frac{dy}{dx} = \frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}} \times (-2x) + \sin^{-1}x + \frac{x}{\sqrt{1-x^2}}$ $= -\frac{x}{\sqrt{1-x^2}} + \sin^{-1}x + \frac{x}{\sqrt{1-x^2}}$ $= \sin^{-1}x$	• 2 – correct solution • 1 – correct derivative(s) for parts of the expression
	(ii)	$\int_0^1 \sin^{-1}x\,dx = \left[\sqrt{1-x^2} + x\sin^{-1}x\right]_0^1$ $= \left(\sqrt{1-1^2} + 1\sin^{-1}1\right) - \left(\sqrt{1-0^2} + 0\sin^{-1}0\right)$ $= \frac{\pi}{2} - 1$	• 1 – correct solution
(c)	(i)	$f(x) = x(x+1) - a(a+1)$ $f(a) = a(a+1) - a(a+1)$ $= 0$ $\therefore (x-a)$ is a factor of $f(x)$.	• 1 – correct solution *NB.* Many candidates did not distinguish between roots / zeros of a function and factors of a function: $x = a$ is a <u>root / zero</u> of $f(x)$ $(x - a)$ is a <u>factor</u> of $f(x)$
	(ii)	$\begin{array}{r\|l} & x + (a+1) \\ \hline x-a & x^2 + x - a(a+1) \\ & \underline{x^2 - ax \qquad\qquad -} \\ & x(1+a) - a(a+1) \\ & \underline{x(a+1) - a(a+1) \ -} \\ & 0 \end{array}$ $\therefore x(x+1) - a(a+1) = (x-a)(x+a+1)$	• 1 – correct solution

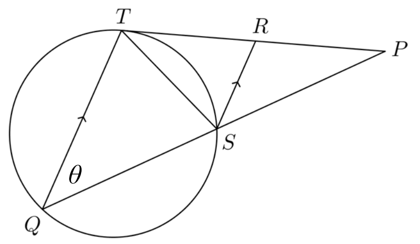
Question 13 (continued)

	Sample solution	Suggested marking criteria
(d)	Let $S(n)$ be the statement that $4^n + 5^n + 6^n$ is divisible by 15. <u>Initial case, $S(1)$:</u> $4^1 + 5^1 + 6^1 = 15$, clearly divisible by 15. $\therefore S(1)$ is true. <u>Assume $S(k)$ is true for some odd integer k:</u> i.e. $4^k + 5^k + 6^k = 15M$, for some integer M. <u>Show $S(k+2)$ is true:</u> i.e. $4^{k+2} + 5^{k+2} + 6^{k+2} = 15N$, for some integer N. $\text{LHS} = 4^{k+2} + 5^{k+2} + 6^{k+2}$ $= 4^2 \times 4^k + 5^2 \times 5^k + 6^2 \times 6^k$ $= 16\left(15M - 5^k - 6^k\right) + 25 \times 5^k + 36 \times 6^k$ $= 15 \times 16M - 16 \times 5^k + 25 \times 5^k - 16 \times 6^k + 36 \times 6^k$ $= 15 \times 16M + 9 \times 5^k + 20 \times 6^k$ $= 15 \times 16M + 45 \times 5^{k-1} + 120 \times 6^{k-1}$ $= 15\left(16M + 3 \times 5^{k-1} + 8 \times 6^{k-1}\right)$ $= 15N$, for some integer $N = 16M + 3 \times 5^{k-1} + 8 \times 6^{k-1}$ $\therefore S(k+2)$ is true if $S(k)$ is assumed true for some odd integer k. Since $S(1)$ is shown true, and $S(k+2)$ is true if $S(k)$ is true, by the principle of mathematical induction, $S(n)$ is true for all odd integers $n \geq 1$.	• 3 – correct solution • 2 – uses the inductive assumption appropriately towards solution • 1 – verifies the initial case *NB. Many candidates neglected to define k.*
(e)	$\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^9 = \sum_{r=0}^{9} {}^9C_r \left(\frac{3x^2}{2}\right)^{9-r} \left(-\frac{1}{3x}\right)^r$ $= \sum_{r=0}^{9} {}^9C_r \left(\frac{3}{2}\right)^{9-r} \left(x^2\right)^{9-r} \left(-\frac{1}{3}\right)^r \left(\frac{1}{x}\right)^r$ $= \sum_{r=0}^{9} {}^9C_r \left(\frac{3^{9-r}}{2^{9-r}}\right) (-1)^r \left(3^{-r}\right) \frac{x^{18-2r}}{x^r}$ $= \sum_{r=0}^{9} {}^9C_r (-1)^r \left(\frac{3^{9-2r}}{2^{9-r}}\right) x^{18-3r}$ $T_{r+1} = {}^9C_r (-1)^r \left(\frac{3^{9-2r}}{2^{9-r}}\right) x^{18-3r}$ Term independent of x when $18 - 3r = 0$, i.e. $r = 6$ $T_7 = {}^9C_6 (-1)^6 \left(\frac{3^{9-2\times 6}}{2^{9-6}}\right)$ $= {}^9C_6 \times \frac{1}{3^3} \times \frac{1}{2^3}$ $= \frac{7}{18}$	• 3 – correct answer / simplified expression • 2 – finds the condition for the constant term • 1 – identifies the general term of the expansion

Question 14

Sample solution			Suggested marking criteria
(a)	(i)	When $\alpha = 30°$: $x = 12t\cos 30° = 6\sqrt{3}\,t$ $y = -5t^2 + 12t\sin 30° - 1 = -5t^2 + 6t - 1$ $y = 0$ when: $-5t^2 + 6t - 1 = 0$ $5t^2 - 6t + 1 = 0$ $5t^2 - 5t - t + 1 = 0$ $5t(t-1) - (t-1) = 0$ $(t-1)(5t-1) = 0$ When $t = 1$, $x = 6\sqrt{3}$, $\therefore$ the golf ball lands $\left(6\sqrt{3} - 4\right)$ metres right of A.	• 2 – correct solution • 1 – finds the total horizontal displacement of the golf ball
	(ii)	$y = -5t^2 + 6t - 1$ $\dot{y} = -10t + 6$ $\dot{y} = 0$ when $-10t + 6 = 0$, ie. $t = \frac{3}{5}$. When $t = \frac{3}{5}$: $y = -5 \times \left(\frac{3}{5}\right)^2 + 6 \times \frac{3}{5} - 1 = \frac{4}{5}$ Therefore, the golf ball reaches a maximum height of $\frac{4}{5}$ metres above level ground.	• 2 – correct solution • 1 – finds the time when vertical velocity is zero
	(iii)	Horizontally, the golf ball must reach A: $4 = 12t\cos\alpha$ $\frac{1}{3\cos\alpha} = t$ $y = -5\left(\frac{1}{3\cos\alpha}\right)^2 + 12\left(\frac{1}{3\cos\alpha}\right)\sin\alpha - 1$ $= -\frac{5}{9}\sec^2\alpha + 4\tan\alpha - 1$ $= -\frac{5}{9}\left(\tan^2\alpha + 1\right) + 4\tan\alpha - 1$ $= -\frac{5}{9}\tan^2\alpha + 4\tan\alpha - \frac{14}{9}$ The golf ball will land on level ground when it satisfies: $-\frac{5}{9}\tan^2\alpha + 4\tan\alpha - \frac{14}{9} = 0$ $5\tan^2\alpha - 36\tan\alpha + 14 = 0$ $\tan\alpha = \frac{-(-36) \pm \sqrt{(-36)^2 - 4 \times 5 \times 14}}{2 \times 5} = \frac{36 \pm \sqrt{1016}}{10}$ $\alpha = 22°25'$ or $81°37'$ Therefore, the particle will land on level ground to the right of A if $22°25' < \alpha < 81°37'$ (nearest minute).	• 3 – correct solution • 2 – attempts to find 2 values of α that satisfy the given condition • 1 – uses $x = 4$ as a boundary value – recognises $y = 0$ for the golf ball to land on level ground

Question 14 (continued)

		Sample solution	Suggested marking criteria
(b)	(i)	 $\angle P$ is common. Let $\angle TQS = \theta$ as shown. $\angle STP = \angle TQS = \theta$ (alternate segment theorem) $\angle RSP = \angle TQS = \theta$ (corresponding angled are equal, $TQ \parallel RS$) $\therefore \angle STP = \angle RSP \ (= \theta)$ $\therefore \Delta PST \ \vert\vert\vert \ \Delta PRS \ (AA)$	• 2 – correct solution • 1 – uses a correct circle geometry theorem in an attempt to show the result
	(ii)	$\frac{PT}{PS} = \frac{ST}{RS}$ (matching sides of similar triangles are in the same ratio) $\therefore PT = \frac{ST \times PS}{RS}$	• 1 – correct solution
(c)	(i)	$\int_0^{\frac{\pi}{2}} (\cos x)^{2k} \sin x \, dx$ $= -\int_0^{\frac{\pi}{2}} (\cos x)^{2k} \times (-\sin x) \, dx$ $= -\int_1^0 u^{2k} \, du$ $= \int_0^1 u^{2k} \, du$ $= \left[\frac{u^{2k+1}}{2k+1} \right]_0^1$ $= \frac{1}{2k+1}$ Let $u = \cos x$ $du = -\sin x \, dx$ When $x = 0, u = 1$ When $x = \frac{\pi}{2}, u = 0$	• 2 – correct solution • 1 – correct primitive in terms of u

Question 14 (continued)

<table>
<tr><th colspan="2">Sample solution</th><th>Suggested marking criteria</th></tr>
<tr><td>(c)</td><td>(ii)
$$\int_0^{\frac{\pi}{2}} (\sin x)^{2n+1}\,dx$$
$$= \int_0^{\frac{\pi}{2}} \sin x\left(1-\cos^2 x\right)^n dx$$
$$= \int_0^{\frac{\pi}{2}} \sin x \sum_{r=0}^{n} {}^nC_r\left(-\cos^2 x\right)^r dx \quad (\text{by Binomial theorem})$$
$$= \int_0^{\frac{\pi}{2}} \sum_{r=0}^{n} (-1)^r\,{}^nC_r \sin x(\cos x)^{2r}\,dx$$
$$= \sum_{r=0}^{n} \int_0^{\frac{\pi}{2}} (-1)^r\,{}^nC_r \sin x(\cos x)^{2r}\,dx \quad (\text{integral of a sum is sum of integrals})$$
$$= \sum_{r=0}^{n} (-1)^r\,{}^nC_r \int_0^{\frac{\pi}{2}} \sin x(\cos x)^{2r}\,dx \quad (\text{constants brought out the front})$$
$$= \sum_{r=0}^{n} (-1)^r\,{}^nC_r \left(\frac{1}{2r+1}\right)$$</td><td>• 2 – correct solution
• 1 – correct use of the binomial theorem to expand $\left(1-\cos^2 x\right)^n$</td></tr>
<tr><td></td><td>(iii) Letting $n = 2$:
$$\int_0^{\frac{\pi}{2}} (\sin x)^5\,dx = \sum_{r=0}^{2} (-1)^r\,{}^2C_r\left(\frac{1}{2r+1}\right)$$
$$= (-1)^0 \times {}^2C_0 \times \frac{1}{1} + (-1)^1 \times {}^2C_1 \times \frac{1}{3} + (-1)^2 \times {}^2C_2 \times \frac{1}{5}$$
$$= \frac{8}{15}$$</td><td>• 1 – correct solution</td></tr>
</table>